# One-parameter families of Jorgensen groups (Hyperbolic Spaces and Related Topics)

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数理解析研究所講究録
1104 巻 1999 年 18-24



# One-parameter families of Jørgensen groups

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ABSTRACT. This paper is a report without proofs on Jørgensen groups obtained recently. Here a Jørgensen group is a Kleinian group whose Jørgensen number is one. In this paper we consider two kinds of one-parameter families of Jørgensen groups. The first family contains the Picard group and the classical modular group and the second one contains the figure-eight knot group.

## 0. Introduction.

It is an important problem to decide whether or not a non-elementary subgroup of the Möbius transformation group, which is denoted by Möb, is discrete. In 1976 Jørgensen [3] gave a necessary condition for a non-elementary Möbius transformation group $G = \langle A, B\rangle$ to be discrete : If $\langle A, B\rangle$ is a non-elementary discrete group, then

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1.$$

The lower bound 1 is best possible.

Let $\langle A, B\rangle$ be a marked two-generator subgroup of Möb. We call

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|$$

the *Jørgensen number* for $\langle A, B\rangle$. Let $G$ be a two-generator subgroup of Möb. The *Jørgensen number* $J(G)$ for the group $G$ is the infimum of $J(A, B)$ where $A$ and $B$ generate $G$. We call a non-elementary two-generator discrete subgroup $G$ of Möb a *Jørgensen group* if $J(G) = 1$.

With respect to Jørgensen numbers it gives rise to the following problems: (1) Problem 1 is to find many Jørgensen groups, (2) Problem 2 is to find the infimum of Jørgensen numbers for some subspaces of the Kleinian space, for example for the Teichmüller space and for the Schottky space. For Problem 1, Jørgensen-Kiikka [4], Jørgensen-Lascurain-Pignataro [5] and Sato-Yamada [12] gave uncountably many non-conjugate Jørgensen groups. For Problem 2, Gilman [2] and Sato [9] gave the

*Partly supported by the Grant-in-Aid for Co-operative Research as well as Scientific Research, the Ministry of Education, Science, Sports and Culture, Japan.

1991 *Mathematics Subject Classification.* Primary 32G15; Secondary 20M10, 30F40.

best lower bound of Jørgensen numbers for purely hyperbolic two-generator groups, and Sato [10], [11] gave the best lower bound of Jørgensen numbers for the classical Schottky space of real type of genus two, $R\mathbf{S}_2$. Namely,

$$\inf\{J(G) \mid G \in R\mathbf{S}_2\} = 4.$$

For the Riley slice, the infimum of Jørgensen numbers is one (see Keen and Series [6] for the definition of the Riley slice).

In this paper we will consider two kinds of one-parameter families of Jørgensen groups. Let $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$ be a group generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix} \quad (\mu \in \mathbf{C}, \sigma \in \mathbf{C} \setminus \{0\}).$$

The first family consists of groups $G_{\mu,\sigma}$ with $\mu = ik$ $(k \in \mathbf{R})$ and $\sigma = 1$, and the second family consists of groups $G_{\mu,\sigma}$ with $\mu = \sqrt{3}i/2$ and $\sigma = -ie^{i\theta}$ $(0 \leq \theta < 2\pi)$. The first family contains the Picard group and the classical modular group, and the second one contains the figure-eight knot group.

This paper contains four theorems. The first theorem is on the first family, most of which are given by Sato-Yamada [12]. The volumes of three fundamental polyhedra in Theorem 1 are new. Theorems 2 through 4 are on the second families.

In §1 we will state some definitions. In §2 the main theorems in this paper will be stated. The proofs of the theorems will appear elsewhere.

After our talk in a seminar of complex analysis at State University of New York (SUNY) at Stony Brook, Professor Maskit pointed out the following: A problem "is the essentially same group as in Theorem 3 discrete ?" was presented by Gehring. Maskit gave an affirmative answer to the problem by extending the group and using Poincaré's polyhedron theorem (see Maskit [7, pp.68-70] for Poincaré's polyhedron theorem). We prove it by finding side pairing transformations of a fundamental polyhedron for the group without extending the group.

Thanks are due to Professor Maskit for many valuable comments and to Professors Kra, Jørgensen, Gilman and Okumura for valuable suggestions. The author also expresses gratitude to Professors Kra and Maskit and Department of Mathematics, SUNY at Stony Brook for hospitality and fine working conditions.

## 1. Definitions

In this section we will state some definitions, for example a Jørgensen group. Let Möb denote the set of all Möbius transformations. In this paper we use a Kleinian group in the same meaning as a discrete group.

THEOREM A (Jørgensen [3]). *Suppose that the Möbius transformations $A$ and $B$ generate a non-elementary discrete group. Then*

$$J(A, B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2| \geq 1,$$

*where* tr *is the trace. The lower bound* 1 *is best possible.*

DEFINITION 1.1. Let $\langle A, B\rangle$ be a marked two-generator subgroup of Möb. The *Jørgensen number* $J(A,B)$ for $\langle A, B\rangle$ is

$$J(A,B) := |\mathrm{tr}^2(A) - 4| + |\mathrm{tr}(ABA^{-1}B^{-1}) - 2|.$$

DEFINITION 1.2. Let $G$ be a non-elementary subgroup of Möb. The *Jørgensen number* $J(G)$ for $G$ is defined as follows:

$J(G) := \inf\{J(A,B) \mid A \textit{ and } B \textit{ generate } G\}.$

DEFINITION 1.3. A non-elementary two-generator subgroup $G$ of Möb is a *Jørgensen group* if $G$ is a discrete group with $J(G) = 1$.

It is obvious by Theorem A and Theorem 5.4.2 in Beardon [1, p.108] that if $G$ is a non-elementary discrete group, then $J(G) \geq 1$. However there are non-discrete two-generator groups $G = \langle A, B\rangle$ such that $J(A,B) = 1$ as seen in Theorem 1(v) or Theorem 4. Jørgensen-Kiikka [4], Jørgensen-Lascurain-Pigmataro [5] and Sato-Yamada [12] studied two-generator groups $\langle A_1, A_2\rangle$ with $J(A_1, A_2) = 1$.

CONJECTURE. Let $G$ be a non-elementary two-generator subgroup of Möb which does not contain elliptic transformations of infinite order. Then $G$ is a discrete group if and only if $J(G) \geq 1$.

This conjecture means that if $J(A,B) = 1$ and $G$ generated by $A$ and $B$ is not a discrete group, then there exist $C, D \in G$ such that $G = \langle C, D\rangle$ and $J(C,D) < 1$.

We think of the transformation

$$g(z) = \frac{az+b}{cz+d} \qquad (ad - bc = 1)$$

as the matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad (ad - bc = 1).$$

Throughout this paper, we will always write elements in Möb as matrices with determinant 1.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we say that

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}$$

is the *complex conjugate* of $A$. Furthermore, if $G = \langle A_1, A_2, \cdots, A_n\rangle$ and $\bar{G} = \langle \bar{A}_1, \cdots, \bar{A}_n\rangle$, then we say that $\bar{G}$ is the *complex conjugate* of $G$.

## 2. Theorems.

We will consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = \begin{pmatrix} \mu\sigma & \mu^2\sigma - 1/\sigma \\ \sigma & \mu\sigma \end{pmatrix} \quad (\mu \in \mathbf{C}, \sigma \in \mathbf{C}\setminus\{0\}).$$

The family of groups $\{\langle A, B_{1/\sigma,\sigma}\rangle | \sigma \in \mathbf{C}\setminus\{0\}\}$ is the Riley slice $RS$. If $\langle A, B_{1/\sigma,\sigma}\rangle$ is a group in the Riley slice, then $J(A, B_{1/\sigma,\sigma}) = |\sigma|^2$. It is easily seen that

$$\inf\{J(G) \mid G \in RS\} = 1,$$

since $J(A, B_{1/\sigma,\sigma}) = 1$ for $\sigma = 1$, that is, in this case the group is the modular group.

In this paper we only consider the case of $\sigma \in \mathbf{C}\setminus\{0\}$ and $\mu = ik$ $(k \in \mathbf{R})$, that is, we consider two-generator groups $G_{\mu,\sigma} = \langle A, B_{\mu,\sigma}\rangle$, where

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_{\mu,\sigma} = B_{ik,\sigma} = \begin{pmatrix} ik\sigma & -k^2\sigma - 1/\sigma \\ \sigma & ik\sigma \end{pmatrix} \quad (k \in \mathbf{R}, \sigma \in \mathbf{C}\setminus\{0\}).$$

PROPOSITION 2.1. *Let $G_{ik,\sigma} = \langle A, B_{ik,\sigma}\rangle$ be the above group. Let $C_1$ and $C_2$ be the following cylinders:*

$C_1 = \{(\sigma, ik) \mid |\sigma| = 1,\ k \in \mathbf{R}\}$,

$C_2 = \{(\sigma, ik) \mid |\sigma| = 2,\ k \in \mathbf{R}\}$.

(i) *For each point inside the cylinder $C_1$, the corresponding group $G_{ik,\sigma}$ is not a Kleinian group.*

(ii) *Let $(\sigma, ik)$ be a point outside of the cylinder $C_2$. If $|k| \geq 1$, then $G_{ik,\sigma}$ is a boundary group of the Schottky space of genus two.*

(iii) Every Jørgensen group of type $G_{ik,\sigma}$ lies on the cylinder $C_1$.

The first family of Jørgensen groups is considered in Sato-Yamada [12].

THEOREM 1 (Sato-Yamada [12]). *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B_{ik,1} = \begin{pmatrix} ik & -(1+k^2) \\ 1 & ik \end{pmatrix} \quad (k \in \mathbf{R})$$

*and let $G_{ik,1} = \langle A, B_{ik,1}\rangle$ be the group generated by $A$ and $B_{ik,1}$. Then the following hold.*

(i) *In the case of $|k| > 1, G_{ik,1}$ is a Jørgensen group, a Kleinian group of the second kind and $\Omega(G_{ik,1})/G_{ik,1}$ is a single Riemann surface with signature* $(0,4;2,2,3,3)$ *for each $k$, where $\Omega(G_{ik,1})$ denotes the region of discontinuity for $G_{ik,1}$.*

(ii) *In the case of $|k| = 1, G_k$ is a Jørgensen group, a Kleinian group of the second kind and $\Omega(G_{ik,1})/G_{ik,1}$ is a single Riemann surface with signature* $(0,3;3,3,\infty)$.

(iii) *In the case of $\sqrt{3}/2 < |k| < 1, G_k$ is a Jørgensen group, a Kleinian group of the second kind and $\Omega(G_{ik,1})/G_{ik,1}$ is a single Riemann surface with signature* $(0,3;3,3,q)$ *for $k$ with $k^2 = \{1 + \cos(\pi/q)\}/2, q = 4, 5, 6, \cdots$.*

(iv) *In the case of* $1/2 \leq |k| \leq \sqrt{3}/2, G_{ik,1}$ *is a Jørgensen group, a Kleinian group of the first kind for* $|k| = \sqrt{3}/2, \sqrt{2}/2$ *or* $1/2$. *The volumes* $V(G_{ik,1})$ *of 3-orbifolds for* $G_{ik,1}$ *are as follows, where* $L(\theta)$ *is the Lobachevskiĭ function:*

$L(\theta) = -\int_0^\theta \log|2\sin u| du.$

(1) $V(G_{i\sqrt{3}/2,1}) = 2\{L(\pi/6) + L(\pi/3)\}.$

(2) $V(G_{i\sqrt{2}/2,1}) = 2\{2\{L(\pi/4) - L(5\pi/12) - L(\pi/12)\}.$

(3) $V(G_{i/2,1}) = L(\pi/6) + 2L(\pi/3) - L(\varphi_0 + \pi/6) + L(\varphi_0 - \pi/6),$
*where* $\varphi_0 = \sin^{-1}(1/2\sqrt{3})$.

(v) *In the case of* $0 < |k| < 1/2, G_{ik,1}$ *is not a Kleinian group for every* $k$.

(vi) *In the case of* $k = 0, G_{ik,1}$ *is a Jørgensen group, a Kleinian group of the second kind and* $\Omega(G_{ik,1})/G_{ik,1}$ *is a union of two Riemann surfaces with signature* $(0, 3; 2, 3, \infty)$.

REMARKS. (1) The group $G_{i/2,1}$ is conjugate to the Picard group in Möb.
(2) The group $G_{0,1}$ is the classical modular group.

COROLLARY. *Let* $G_{ik,1}$ $(k \in \mathbf{R})$ *be as in Theorem 1. Then* $G_{ik,1}$ *is a Jørgensen group when* $|k| \geq 1$, $k^2 = \{1 + \cos(\pi/q)\}/2$ $(q = 4, 5, 6, \cdots)$, $|k| = \sqrt{3}/2, \sqrt{2}/2, 1/2$ *or* $k = 0$.

Next we will consider the second one-parameter family which consists of groups $G_{\mu,\sigma}$ with $\mu = \sqrt{3}i/2$ and $\sigma = -ie^{i\theta}$ $(0 \leq \theta < 2\pi)$. For simplicity, we set $B_\theta := B_{\sqrt{3}i/2, -ie^{i\theta}}$ and $G_\theta := G_{\sqrt{3}i/2, -ie^{i\theta}}$, that is,

$$B_\theta = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

and $G_\theta = \langle A, B_\theta \rangle$.

LEMMA 2.2. *Let* $B_\theta$ $(0 \leq \theta \leq \pi/2)$ *be the above matrix, and let* $\bar{B}_\theta$ *be the complex conjugate of* $B_\theta$. *Then* $B_{\pi-\theta} = -\bar{B}_\theta^{-1}$.

LEMMA 2.3. *Let* $B_\theta$ *and* $G_\theta$ *be the above. Then* $B_{\pi+\theta} = B_\theta$ *and* $G_{\pi+\theta} = G_\theta$.

By Lemmas 2.2 and 2.3 it suffices to consider $G_{\mu,\sigma}$ with $\mu = \sqrt{3}i/2$ and $\sigma = -ie^{i\theta}$ $(0 \leq \theta \leq \pi/2)$.

THEOREM 2. *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B_\theta := B_{\sqrt{3}i/2, -ie^{i\theta}} = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

*and let* $G_\theta = \langle A, B_\theta \rangle$ *be the group generated by* $A$ *and* $B_\theta$ $(0 \leq \theta \leq \pi)$. *Then*

(i) *In the case of* $\theta = \pi/6$, $G_{\pi/6}$ *is conjugate to the figure-eight knot group and has the following properties:*

(1) $G_{\pi/6}$ *is a Kleinian group of the first kind.*

(2) *$G_{\pi/6}$ is a Jørgensen group.*
(3) $V(G_{\pi/6}) = 6L(\pi/3)$, *where $L(\theta)$ is the Lobachevskiĭ function.*

(ii) *In the case of $\theta = \pi/2$, $G_{\pi/2}$ has the following properties:*
(1) *$G_{\pi/2}$ is a Kleinian group of the first kind.*
(2) *$G_{\pi/2}$ is a Jørgensen group.*
(3) $V(G_{\pi/2}) = 2\{L(\pi/6) + L(\pi/3)\}$.

(iii) *In the case of $\theta = 5\pi/6$, $G_{5\pi/6}$ is the complex conjugate to the figure-eight knot group and has the following properties:*
(1) *$G_{5\pi/6}$ is a Kleinian group of the first kind.*
(2) *$G_{5\pi/6}$ is a Jørgensen group.*
(3) $V(G_{5\pi/6}) = 6L(\pi/3)$.

THEOREM 3 (Maskit [8]). *In the case of $\theta = 0$, that is, in the case where*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_0 = \begin{pmatrix} \sqrt{3}/2 & -i/4 \\ -i & \sqrt{3}/2 \end{pmatrix},$$

*$G_0 = \langle A, B_0 \rangle$ has the following properties :*
(1) *$G_0$ is a Kleinian group of the second kind.*
(2) *$G_0$ is a Jørgensen group.*
(3) *$\Omega(G_0)/G_0$ is a single Riemann surface with signature* $(0, 3; 2, 3, \infty)$.

REMARK. Maskit [8] shows that the essentially same group as $G_0$ is discrete, that is, he shows that a group conjugate to $G_\pi$ is discrete by extending the group and using Poincaré's polyhedron theorem. Our proof is different from his. We prove this theorem by finding side pairing transformations of a fundamental polyhedron for the group $G_0$ and using Poincaré's polyhedron theorem without extending the group.

THEOREM 4. *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_\theta = \begin{pmatrix} \sqrt{3}e^{i\theta}/2 & i(3e^{i\theta}/4 - e^{-i\theta}) \\ -ie^{i\theta} & \sqrt{3}e^{i\theta}/2 \end{pmatrix}$$

*and let $G_\theta = \langle A, B_\theta \rangle$ be the group generated by $A$ and $B_\theta$. If $\theta$ $(0 < \theta \leq \pi/2)$ satisfies the inequality*

$$|e^{2i\theta} - 1||3/2 - e^{2i\theta}| < 1,$$

*then $G_\theta$ is a non-elementary group but not a Kleinian group.*

REMARK. The angle $\theta$ satisfying the inequality in Theorem 4 is $0 < \theta < \theta_0$ $(\theta_0 < \pi/6)$, where $\theta_0$ is very close to $\pi/6$.

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